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Direct Sum and Hyperbolic Complexification of Hopf Algebras

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The problem of how to obtain a hyperbolic complexification of a Hopf algebra from two known Hopf algebras is discussed. We prove that this Hopf algebra is isomorphic to a direct sum of the two known Hopf algebras, and that therefore this direct sum is also a Hopf algebra. In particular, the result can be applied to quantum enveloping algebras.

We have proved (Zhong, 1992), in term of R-matrices, that the product of two quantum linear groups is isomorphic to a hyperbolic complexification of the quantum linear groups. This result is an extension of the results in Zhong (1985) to the quantum linear groups. Therefore we expect that there are some general results for the quantum enveloping algebras. In this paper a general result concerning Hopf algebras is given. Since a quantum enveloping algebra is a noncommutative and noncocommutative Hopf algebra (Jimbo, 1985, 1986), the case of the quantum enveloping algebras is also included.

In the first place, we meet a problem: What sum of two Hopf algebras is still a Hopf algebra? Suppose that A and B are any two Hopf algebras, they are two subalgebras of a k-algebra (Abe, 1977) $\mathscr{A}$ with the unit element, and the real field $R^1 \subset k$. If in the former there is not this $\mathscr{A}$, then we can generate it by using the free products of A and B. Let the Hopf algebra structures of A and B, respectively, consist of the multiplications m_A, m_B, the comultiplications Δ_A, Δ_B, the antipodes γ_A, γ_B, and the counits e_A, e_B as

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$$
\begin{aligned}
\Delta_J\colon &\quad J \to J \otimes J, \qquad \Delta_J(jj') = \Delta_J(j)\Delta_J(j') \\
\gamma_J\colon &\quad J \to J, \qquad \gamma_J(jj') = \gamma_J(j')\gamma_J(j) \\
e_J\colon &\quad J \to \kappa, \qquad e_J(jj') = e_J(j)e_J(j') \\
&j, j' \in J, \qquad J = A, B, \qquad j = a, b
\end{aligned}
\tag{1}
$$

They obey the following relations:

$$
\begin{aligned}
&[\mathrm{id} \otimes \Delta_J]\Delta_J(j) = [\Delta_J \otimes \mathrm{id}]\Delta_J(j) \\
m_J&[\mathrm{id} \otimes \gamma_J]\Delta_J(j) = m_J[\gamma_J \otimes \mathrm{id}]\Delta_J(j) = e_J(j) \cdot \mathbb{1} \\
&[e_J \otimes \mathrm{id}]\Delta_J(j) = [\mathrm{id} \otimes e_J]\Delta_j(j) = j \\
&j \in J, \qquad J = A, B, \qquad j = a, b
\end{aligned}
\tag{2}
$$

In the following the ordinary direct sum of algebras A and B is written as $A \oplus B$, in which an element is written as (a, b) for $a \in A$, $b \in B$. Notice that since $(A \oplus B) \otimes (A \oplus B) \neq (A \otimes A) \oplus (B \otimes B)$, we generally cannot directly obtain a Hopf algebra structure on $S = A \oplus B$ by using the Hopf algebra structure of A and B. However, we can define a set of maps as follows. First we define a revised tensor product $\overline{\otimes}$ as

$$
\begin{aligned}
(a, b)\, \overline{\otimes}\, (a', b') &= (a \otimes a', b \otimes b') \\
a, a' \in A, &\qquad b, b' \in B
\end{aligned}
\tag{3}
$$

In the following according to Abe (1977), we write

$$
\Delta_A(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}, \qquad \Delta_B(b) = \sum_{(b)} b_{(1)} \otimes b_{(2)}, \qquad a \in A, \quad b \in B \tag{4}
$$

Next, let the maps m, Δ, γ, and e, respectively, be defined as

$$
\begin{aligned}
m\colon &\quad S \,\overline{\otimes}\, S \to S, \qquad m[(a, b)\, \overline{\otimes}\, (a', b')] = [m_A(a \otimes a'), m_B(b \otimes b')] \\
\Delta\colon &\quad S \to S \,\overline{\otimes}\, S, \qquad \Delta(a, b) = \sum_{(a),(b)} (a_{(1)}, b_{(1)})\, \overline{\otimes}\, (a_{(2)}, b_{(2)}) \\
\gamma\colon &\quad S \to S, \qquad \gamma(a, b) = (\gamma_A(a), \gamma_B(B)) \\
e\colon &\quad S \to k \times k, \qquad e(a, b) = (e_A(a), e_B(b)) \\
&a, a' \in A, \qquad b, b' \in B
\end{aligned}
\tag{5}
$$

The following relations can be directly verified:

$$\Delta(\alpha\beta) = \Delta(\alpha)\Delta(\beta), \qquad \gamma(\alpha\beta) = \gamma(\beta)\gamma(\alpha), \qquad e(\alpha\beta) = e(\alpha)e(\beta)$$
$$[\mathrm{id}\,\overline{\otimes}\,\Delta]\Delta(\alpha) = [\Delta\,\overline{\otimes}\,\mathrm{id}]\Delta(\alpha)$$
$$m[\mathrm{id}\,\overline{\otimes}\,\gamma]\Delta(\alpha) = m[\gamma\,\overline{\otimes}\,\mathrm{id}]\Delta(\alpha) = e(\alpha)\cdot\mathbb{1} \tag{6}$$
$$[\mathrm{id}\,\overline{\otimes}\,e]\Delta(\alpha) = [e\,\overline{\otimes}\,\mathrm{id}]\Delta(\alpha) = \alpha$$
$$\alpha, \beta \in S$$

Therefore we obtain a structure $\{S, m, \Delta, \gamma, e\}$, which is very like a Hopf algebra. We call this structure the *direct sum* of the Hopf algebras A and B, and write it as $A \oplus_H B$. Since $\overline{\otimes}$ is not a proper tensor product on S, temporarily we are not sure whether $A \oplus_H B$ is a Hopf algebra. However, in the following we prove that $A \oplus_H B$ is isomorphic to a Hopf algebra whose elements have hyperbolic complexified forms; then $A \oplus_H B$, in fact, can be also regarded a Hopf algebra.

For the hyperbolic number ring and its application in Lie group theory, see Zhong (1985) and Yaglom (1979). Let ε denote the hyperbolic pure imaginary unit, $\varepsilon^2 = +1$, $\varepsilon \neq \pm 1$. If x and y are real numbers, then the set of all hyperbolic numbers $x + \varepsilon \cdot y$ forms a commutative ring H. A characteristic of the ring H is that there are mutually nullification divisors (Zhong, 1985) z and z^*,

$$z = \frac{1+\varepsilon}{2}, \qquad z^* = \frac{1-\varepsilon}{2}, \qquad zz^* = 0, \qquad z^2 = z, \qquad z^{*2} = z^* \tag{7}$$

In an H-algebra $\mathcal{B}$ there must be the following identity:

$$\left[\frac{1}{2}(r+s) + \frac{\varepsilon}{2}(r-s)\right]\left[\frac{1}{2}(r'+s') + \frac{\varepsilon}{2}(r'-s')\right]$$
$$= \frac{1}{2}(rr' + ss') + \frac{\varepsilon}{2}(rr' - ss'), \qquad r, r', s, s' \in \mathcal{B} \tag{8}$$

Now we construct a Hopf algebra $\mathcal{H}_{(A,B)}$ from the Hopf algebras A and B as follows. Define that ε is commutative with any element of the k-algebra $\mathcal{A}$. Let $\mathcal{H}_{(A,B)}$ be the set of all elements of the form

$$h = \frac{a+b}{2} + \varepsilon\frac{a-b}{2}, \qquad a \in A, \quad b \in B \tag{9}$$

If $w = x + \varepsilon \cdot y$ is an arbitrary hyperbolic complex number, where x and y are real numbers, then

$$wh = \frac{\tilde{a} + \tilde{b}}{2} + \varepsilon \frac{\tilde{a} - \tilde{b}}{2}, \qquad \tilde{a} = (x + y)a \in A, \qquad \tilde{b} = (x - y)b \in B \tag{10}$$

Thus it is easily seen by using Eqs. (8) and (10), that $\mathcal{H}_{(A,B)}$ is an algebra over the commutative ring H. For $h = \frac{1}{2}(a + b) + \frac{1}{2}\varepsilon(a - b)$, $h' = \frac{1}{2}(a' + b') + \frac{1}{2}\varepsilon(a' - b')$, $a, a' \in A$, $b, b' \in B$, the multiplication m_H in $\mathcal{H}_{(A,B)}$ is

$$m_H(h \otimes h') = hh' = \frac{1}{2}[m_A(a \otimes a') + m_B(b \otimes b')] + \frac{\varepsilon}{2}[m_A(a \otimes a') - m_B(b \otimes b')] \tag{11}$$

Let the operators Δ_H, γ_H, and e_H be defined by

$$\begin{aligned}
\Delta_H:&\quad \mathcal{H}_{(A,B)} \to \mathcal{H}_{(A,B)} \otimes \mathcal{H}_{(A,B)} \\
&\quad \Delta_H(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)} = \frac{1}{2}[\Delta_A(a) + \Delta_B(b)] + \frac{\varepsilon}{2}[\Delta_A(a) - \Delta_B(b)] \\
&\quad h_{(i)} = \frac{1}{2}(a_{(i)} + b_{(i)}) + \frac{\varepsilon}{2}(a_{(i)} - b_{(i)}), \qquad i = 1, 2 \\
\gamma_H:&\quad \mathcal{H}_{(A,B)} \to \mathcal{H}_{(A,B)} \\
&\quad \gamma_H(h) = \frac{1}{2}[\gamma_A(a) + \gamma_B(b)] + \frac{\varepsilon}{2}[\gamma_A(a) - \gamma_B(b)] \\
e_H:&\quad \mathcal{H}_{(A,B)} \to H \\
&\quad e_H(h) = \frac{1}{2}[e_A(a) + e_B(b)] + \frac{\varepsilon}{2}[e_A(a) - e_B(b)]
\end{aligned} \tag{12}$$

where $a_{(i)}$ and $b_{(i)}$ $(i = 1, 2)$ are defined by Eq. (4). Using Eqs. (8)–(12), we can prove that the following relations hold:

$$\begin{aligned}
\Delta_H(hh') &= \Delta_H(h)\Delta_H(h'), \qquad \gamma_H(hh') = \gamma_H(h')\gamma_H(h), \\
e_H(hh') &= e_H(h)e_H(h') \\
[\mathrm{id} \otimes \Delta_H]\Delta_H(h) &= [\Delta_H \otimes \mathrm{id}]\Delta_H(h) \\
m_H[\mathrm{id} \otimes \gamma_H]\Delta_H(h) &= m_H[\gamma_H \otimes \mathrm{id}]\Delta_H(h) = e_H(h) \cdot \mathbb{1} \\
[\mathrm{id} \otimes e_H]\Delta_H(h) &= [e_H \otimes \mathrm{id}]\Delta_H(h) = h
\end{aligned} \tag{13}$$

For example, we have

$$
\begin{aligned}
&[\mathrm{id} \otimes \Delta_H]\Delta_H(h) \\
&\quad = [\mathrm{id} \otimes \Delta_H] \sum_{(h)} h_{(1)} \otimes h_{(2)} = \sum_{(h)} h_{(1)} \otimes \Delta_H(h_{(2)}) \\
&\quad = \sum_{(a),(b)} \left\{ \frac{1}{2} [a_{(1)} \otimes \Delta_A(a_{(2)}) + b_{(1)} \otimes \Delta_B(b_{(2)})] \right. \\
&\qquad \left. + \frac{\varepsilon}{2} [a_{(1)} \otimes \Delta_A(a_{(2)}) - b_{(1)} \otimes \Delta_B(b_{(2)})] \right\} \\
&\quad = \frac{1}{2} [(\mathrm{id} \otimes \Delta_A)\Delta_A(a) + (\mathrm{id} \otimes \Delta_B)\Delta_B(b)] \qquad (14) \\
&\qquad + \frac{\varepsilon}{2} [(\mathrm{id} \otimes \Delta_A)\Delta_A(a) - (\mathrm{id} \otimes \Delta_B)\Delta_B(b)] \\
&\quad = \frac{1}{2} [(\Delta_A \otimes \mathrm{id})\Delta_A(a) + (\Delta_B \otimes \mathrm{id})\Delta_B(b)] \\
&\qquad + \frac{\varepsilon}{2} [(\Delta_A \otimes \mathrm{id})\Delta_A(a) - (\Delta_B \otimes \mathrm{id})\Delta_B(b)] \\
&\quad = [\Delta_H \otimes \mathrm{id}]\Delta_H(h)
\end{aligned}
$$

etc. Therefore $\mathcal{H}_{(A,B)}$, indeed, is a Hopf algebra over H with the special hyperbolic complexification elements as in Eq. (9).

Now we consider the relation between $A \oplus_H B$ and $\mathcal{H}_{(A,B)}$. Let the map ρ be defined by

$$\rho: \quad A \oplus_H B \to \mathcal{H}_{(A,B)}$$

$$\rho(a, b) = \frac{1}{2}(a + b) + \frac{\varepsilon}{2}(a - b), \qquad a \in A, \quad b \in B \qquad (15)$$

Then the following relations are easily verified:

$$
\begin{aligned}
\rho(g\alpha + l\beta) &= g\rho(\alpha) + l\rho(\beta) \\
\rho(\alpha\beta) &= \rho(\alpha)\rho(\beta) \\
\rho[m(\alpha \bar{\otimes} \beta)] &= m_H[(\rho(\alpha) \otimes \rho(\beta)] \\
[\rho \bar{\otimes} \rho]\Delta(\alpha) &= \Delta_H[\rho(\alpha)] \qquad (16) \\
\rho[\gamma(\alpha)] &= \gamma_H[\rho(\alpha)] \\
\rho[e(\alpha)] &= e_H[\rho(\alpha)] \\
\alpha, \beta \in S, &\qquad g, l \in K
\end{aligned}
$$

This means that ρ is an isomorphic relation

$$\rho: \quad A \oplus_H B \overset{\text{Hopf}}{\approx} \mathcal{H}_{(A,B)} \tag{17}$$

Thus $A \oplus_H B$ is also a Hopf algebra, but its explicit Hopf algebra structure appears in $\mathcal{H}_{(A,B)}$; this is very interesting.

An important case is $A = B$, for which

$$A \oplus_H A \overset{\text{Hopf}}{\approx} A_H = \{h = w|a \in A, w \in H\} \tag{18}$$

$$\Delta_H = \Delta_A, \qquad \gamma_H = \gamma_A, \qquad e_H = e_A \qquad [e_H(\varepsilon) = \varepsilon]$$

where A_H is just the hyperbolic complexification of the algebra A.

The above results are extensions of Zhong (1992). They can be applied to the quantum enveloping algebras, i.e., if $A = U_q(L)$ and $B = U_q(L')$, where L and L' are Lie algebras, then $\mathcal{H}_{(A,B)} \overset{\text{Hopf}}{\approx} A \oplus_H B$ is also a quantum algebra, though it is different from the quantum enveloping algebra $U_q(L \oplus L')$.

Example. If $A = B = U_q[su(2)]$, then

$$U_q[su(2)] \oplus_H U_q[su(2)] \overset{\text{Hopf}}{\approx} U_q[su(2);\, H] \tag{19}$$

The algebra $U_q[su(2);\, H]$ is generated by the operators $\mathbb{1}$, ε, $J_3^{(q)}$, and $J_\pm^{(q)}$, where $J_3^{(q)}$, and $J_\pm^{(q)}$, are just the ordinary generating operators of the quantum enveloping algebra $U_q[su(2)]$(q is not a root of unity)

$$[J_3^{(q)}, J_\pm^{(q)}] = \pm J_\pm^{(q)}, \qquad [J_+^{(q)}, J_-^{(q)}] = [2J_3^{(q)}] = \frac{q^{2J_3^{(q)}} - q^{-2J_3^{(q)}}}{q^2 - q^{-2}} \tag{20}$$

The Hopf algebra structure of $U_q[su(2);\, H]$ it consists of

$$\begin{gathered} \Delta(J_3^{(q)}) = J_3^{(q)} \otimes \mathbb{1} + \mathbb{1} \otimes J_3^{(q)} \\ \Delta(J_\pm^{(q)} = J_\pm^{(q)} \otimes q^{1/2J_3^{(q)}} + q^{-1/2J_3^{(q)}} \otimes J_\pm^{(q)} \\ e(\mathbb{1}) = 1, \qquad e(\varepsilon) = \varepsilon, \qquad e(J_3^{(q)}) = e(J_\pm^{(q)}) = 0 \\ \gamma(J_3^{(q)}) = -J_3^{(q)}, \qquad \gamma(J_\pm^{(q)}) = -q^{\mp 1/2} J_\pm^{(q)} \end{gathered} \tag{21}$$

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